Gluing semiclassical resolvent estimates, or the importance of being microlocal

András Vasy
(joint work with Kiril Datchev)

In this talk I give a method for gluing high energy or semiclassical resolvent estimates, i.e. to obtain global resolvent estimates when analogous estimates are known for local models. Such a method is useful because the estimates for the local models can be obtained using different techniques, which might not be easy to combine directly. The key point is the use of a microlocal understanding of the propagation of semiclassical singularities to patch the resolvents.

As an application, one can describe solutions of the wave equation modulo exponential decay when

(1) One has a model at infinity with good high energy resolvent estimates, such as asymptotically hyperbolic spaces, see the work of Melrose, Sá Barreto and the lecturer [7].
(2) One has 'mild' trapping in a compact set, such as normally hyperbolic trapped sets, see the recent work of Wunsch and Zworski [10].

This combination gives a *more robust* way of analyzing wave propagation on de Sitter-Schwarzschild space than done earlier in [7, 6], which relied on combining the first listed ingredient with high energy estimates for the cutoff resolvent (i.e. for the actual resolvent on the whole of de Sitter-Schwarzschild space, sandwiched between cutoffs, due to Bony and Häfner [1], who used one dimensional techniques) by the technique of Bruneau and Petkov [2].

We actually work with more general semiclassical resolvent estimates which can be motivated as follows. Let $\tilde{P} = \Delta_g + V$ on a Riemannian manifold (X, g) with a real potential V, and let $R(\lambda) = (\tilde{P} - \lambda)^{-1}$ be the resolvent of $\tilde{P}$ when $\operatorname{Im}\lambda > 0$, as well as its analytic continuation across the positive real axis $(\lambda_0, +\infty)$, λ_0 sufficiently large, when this exists. A contour deformation (in τ, the dual of t) argument shows that, as far as high energy behavior is considered, it suffices to obtain polynomial (in $\operatorname{Re}\tau$) estimates for $R(\tau^2)$, $|\operatorname{Im}\tau| < \Gamma'$, $\Gamma < \Gamma'$, in order to understand the solutions of the wave equation modulo exponential decay $e^{-\Gamma t}$, in spatially compact sets. With $h = |\operatorname{Re}\tau|^{-1}$, and rewriting τ, one is left to consider $h^{-2}(h^2(\Delta + V) - 1 - z)$, with $|\operatorname{Im} z| \leq Ch$ (and $\operatorname{Re} z$ is $\mathcal{O}(h^2)$).

The semiclassical principal symbol of the more general operator $P = h^2\Delta_g + V - 1$ is $p = |\xi|_g^2 + V(x) - 1$, which thus vanishes on the typically non-empty characteristic set $\Sigma_p = \{(x, \xi) :\ p(x, \xi) = 0\}$, so even though the standard principal symbol of P is elliptic, P is *not* elliptic in the semiclassical sense. From the local perspective, the best case scenario is if p is real principal type, i.e. the Hamilton vector field H_p does not vanish on Σ_p. This is analogous in the standard ps.d.o. world to (micro)hyperbolic equations, such as the wave equation, where one has the loss of one order of derivative relative to the elliptic case. Correspondingly, one may hope for estimates such as $\|(P - z)^{-1}\| \leq Ch^{-1}$, with the norm being as an operator acting on some weighted spaces. These indeed hold in asymptotically hyperbolic spaces, see [7], acting on optimally weighted spaces. In smaller than $\mathcal{O}(h)$ neighborhood of the real axis ($\operatorname{Im} z = 0$), such estimates hold if (X, g) is a non-trapping asymptotically Euclidean, or rather scattering, space, as proved by the lecturer and Zworski [9], as well as in more general geometries as shown by Cardoso and Vodev [3].

The semiclassical wave front set, WF_h, of a function u, measures microlocally, i.e. in T^*X, whether u rapidly decays in h relative to some space (here, L^2), see e.g. [5]. Then real principal type propagation of singularities is the following: Suppose that $u \in h^{-N}L^2$. Then $\mathrm{WF}_h(u) \setminus \mathrm{WF}_h(Pu)$ is a union of maximally extended nullbicharacteristics. Note that even if $Pu = 0$, this allows for $\mathrm{WF}_h(u)$ to be non-empty, much like solutions of the wave equations need not be smooth.

When one is considering a limit such as $R(z)$, with $\operatorname{Im} z \to 0$, for which one has an elliptic problem in $\operatorname{Im} z > 0$, one can sometimes get a one-sided estimate: if the *backward* bicharacteristic from (y, η) is disjoint from $\mathrm{WF}_h(Pu)$, and Pu is compactly supported, say, then $(y, \eta) \notin \mathrm{WF}_h(u)$. Thus, singularities propagate *forwards*. This holds, for instance, on asymptotically hyperbolic spaces, as follows from [7]. In other words, singularities do not appear 'out of nowhere' from $-\infty$ along bicharacteristics. The same holds for solutions of operators of the form $P - iW$, at least microlocally along bicharacteristics that reach $T^*W^{-1}(1)$ in finite time, where $W \in C^\infty(X_1'; [0, 1])$ has $W = 0$ on X_1 and $W = 1$ off a compact set, see the work of Nonnenmacher and Zworski [8]. In fact, complex absorbing potentials provide a convenient way of localizing problems to trapped sets, see e.g. [10, 4], so our gluing construction is expected to be very useful in applications.

To set up the gluing problem, suppose $\bar{X}$ is a compact manifold with boundary, X its interior, x a boundary defining function, (X, g) is complete, $P = h^2\Delta_g + V - 1$ is self-adjoint. Let $X_0 = \{x < 4\}$, $X_1 = \{x > 1\}$. The first serious assumption is that level sets of x are (null)bicharacteristically convex in the overlap $X_0 \cap X_1$, i.e. if γ is a nullbicharacteristic then $\dot{x}(\gamma(t)) = 0$ implies $\ddot{x}(\gamma(t)) < 0$. This states $x \circ \gamma$ can only have strict local maxima as critical points. It is this convexity that will assure that the iterative construction we give ends in finitely many (three) steps.

Next, we assume that there are manifolds X_j', $j = 0, 1$, including X_j as open sets, with some not necessarily self-adjoint semiclassical Schrödinger operators P_j, such that $P_j|_{X_j} = P|_{X_j}$. We also assume that X_1 is bicharacteristically convex for P_1, i.e. that no (null)bicharacteristic of P_1 can leave X_1 and return there; this holds in most cases of interest. Assume also that the resolvents $R_j(z)$ extend analytically to some set $D \subset [-E, E] + i[-Ch, Ch]$, and, acting on certain weighted spaces, with weight non-vanishing in $X_0 \cap X_1$ for R_0 and in X_1 for R_1, satisfy polynomial bounds $\|R_j(z)\| \leq a_j(h) \leq h^{-N}$, for $0 < h \leq h_0$ and some N.

The most important assumption is a microlocal one on the P_j. Suppose $q \in T^*X_j'$ is in the characteristic set of P_j, and let $\gamma_- : (-\infty, 0] \to T^*X_j'$ be the backward P_j-bicharacteristic from q. We say that the resolvent $R_j(z)$ is *semiclassically outgoing at q* if $u \in L^2_{\mathrm{comp}}(X_j)$ polynomially bounded, $\mathrm{WF}_h(u) \cap \gamma_- = \emptyset$ implies that $q \notin \mathrm{WF}_h(R_j(z)u)$, i.e. WF_h could only arise from the past of q. Our microlocal assumption is then that

(0-OG) $R_0(z)$ is semiclassically outgoing at all $q \in T^*(X_0 \cap X_1) \cap \Sigma_p$,

(1-OG) $R_1(z)$ is semiclassically outgoing at all $q \in T^*(X_0 \cap X_1) \cap \Sigma_p$ such that γ_- is disjoint from $T^*(X_1' \setminus (X \setminus X_0))$, thus disjoint from any trapping in X_1.

Theorem 1. *There exists $h_0 \in (0,1)$ such that for $h < h_0$, $R(z)$ continues analytically to D and obeys the bound $\|R(z)\| \leq Ch^2 a_0^2 a_1$ there, with the norm taken in the same weighted space as for $R_0(z)$.*

In particular, when $a_0 = C/h$, we find that $R(z)$ obeys (up to constant factor) the same bound as $R_1(z)$, the model operator with infinity suppressed.

In order to prove the theorem, we construct a semiclassical parametrix. Let $\chi_1 \in C_0^\infty(X;[0,1])$ be such that $\chi_1 = 1$ near $\{x \geq 3\}$ and $\operatorname{supp} \chi_1 \subset \{x > 2\}$ and let $\chi_0 = 1 - \chi_1$. Define a right parametrix for P by

$$F \equiv \chi_0(x-1)R_0(z)\chi_0(x) + \chi_1(x+1)R_1(z)\chi_1, \text{ so}$$

$$PF = \operatorname{Id} + [P, \chi_0(x-1)]R_0(z)\chi_0 + [P, \chi_1(x+1)]R_1(z)\chi_1 \equiv \operatorname{Id} + A_0 + A_1.$$

The error $A_0 + A_1$ is large, $\mathcal{O}(1)$, in h due to semiclassical propagation of singularities, but using an iteration argument we can replace it by a small error.

The key point is that by the *forward propagation of semiclassical singularities*, i.e. the outgoing assumptions on the resolvent, $\|A_0 A_1\|_{L^2 \to L^2} = \mathcal{O}(h^\infty)$. Indeed, for a pair of points to be in the wave front relation of the product, there must be a nullbicharacteristic of P going through three points in T^*X over $\operatorname{supp} \chi_1$, $\operatorname{supp} d\chi_1(.+1)$ and $\operatorname{supp} d\chi_0(.-1)$ in this order, which is excluded by the convexity assumption. This implies that iterating the parametrix construction, i.e. solving away the A_0 error using R_1 and solving away the A_1 error using R_0, and repeating once more, the error is $\mathcal{O}(h^\infty)$.

References

[1] J.-F. Bony and D. Häfner. *Decay and non-decay of the local energy for the wave equation in the De Sitter - Schwarzschild metric*, Comm. Math. Phys. **282** (2008), 697–719.

[2] V. Bruneau and V. Petkov. *Semiclassical Resolvent Estimates for Trapping Perturbations*, Comm. Math. Phys., **213** (2000), 413–432.

[3] F. Cardoso and G. Vodev. *Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds. II*, Ann. Henri Poincaré, **3** (2002), 673–691.

[4] K. Datchev. *Local smoothing for scattering manifolds with hyperbolic trapped sets*, Comm. Math. Phys., **286** (2009), 837–850.

[5] M. Dimassi and J. Sjöstrand. Spectral asymptotics in the semiclassical limit. *London Math. Soc. Lecture Note Ser.* 268, 1999.

[6] R. B. Melrose, A. Sá Barreto, and A. Vasy. *Asymptotics of solutions of the wave equation on de Sitter-Schwarzschild space.* Preprint available at arXiv:0811.2229.

[7] R. B. Melrose, A. Sá Barreto, and A. Vasy. *Analytic continuation and semiclassical resolvent estimates on asymptotically hyperbolic spaces.* Work in progress, 2010.

[8] S. Nonnenmacher and M. Zworski. *Quantum decay rates in chaotic scattering.* Acta Math., **203** (2009), 149–233.

[9] A. Vasy and M. Zworski. *Semiclassical estimates in asymptotically Euclidean scattering*, Comm. Math. Phys., **212** (2000), 205–217.

[10] J. Wunsch and M. Zworski. *Resolvent estimates for normally hyperbolic trapped sets.* Preprint available at arXiv:1003.4640, 2010.